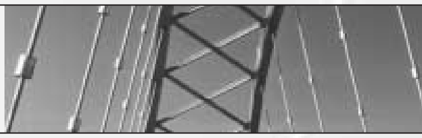


10

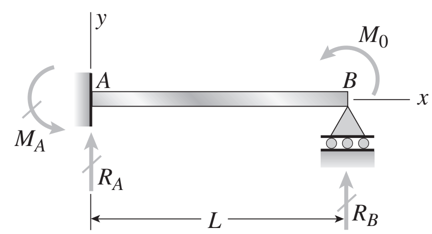
Statically Indeterminate Beams

Differential Equations of the Deflection Curve

The problems for Section 10.3 are to be solved by integrating the differential equations of the deflection curve. All beams have constant flexural rigidity EI. When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.

Problem 10.3-1 A propped cantilever beam AB of length L is loaded by a counterclockwise moment M_0 acting at support B (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.3-1 Propped cantilever beam

M_0 = applied load

Select M_A as the redundant reaction.

REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{M_A}{L} + \frac{M_0}{L} \quad (1) \qquad\qquad R_B = -R_A \quad (2)$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L} \quad (3)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = \frac{M_A}{L}(x - L) + \frac{M_0 x}{L}$$

$$EIv' = \frac{M_A}{L}\left(\frac{x^2}{2} - Lx\right) + \frac{M_0 x^2}{2L} + C_1 \quad (4)$$

B.C. 1 $\quad v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = \frac{M_A}{L}\left(\frac{x^3}{6} - \frac{Lx^2}{2}\right) + \frac{M_0 x^3}{6L} + C_2 \quad (5)$$

B.C. 2 $\quad v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $\quad v(L) = 0 \quad \therefore M_A = \frac{M_0}{2}$

REACTIONS (SEE EQS. 1 AND 2)

$$M_A = \frac{M_0}{2} \qquad R_A = \frac{3M_0}{2L} \qquad R_B = -\frac{3M_0}{2L} \quad \longleftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A = \frac{3M_0}{2L} \quad \longleftarrow$$

BENDING MOMENT (FROM EQ. 3)

$$M = \frac{M_0}{2L}(3x - L) \quad \longleftarrow$$

(Continued)

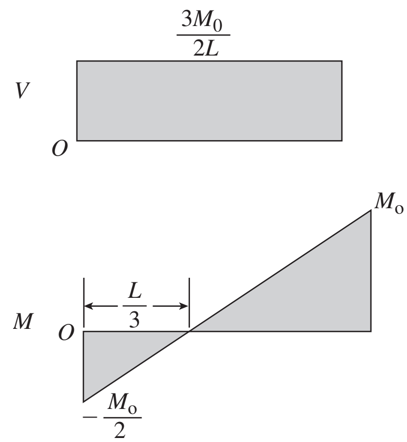
SLOPE (FROM EQ. 4)

$$v' = -\frac{M_0 x}{4LEI}(2L - 3x) \quad \longleftarrow$$

DEFLECTION (FROM EQ. 5)

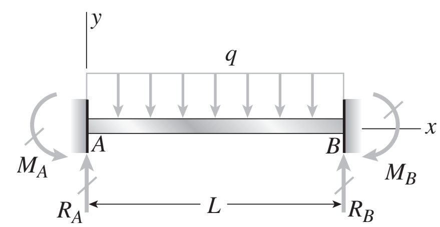
$$v = -\frac{M_0 x^2}{4LEI}(L - x) \quad \longleftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS



Problem 10.3-2 A fixed-end beam AB of length L supports a uniform load of intensity q (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), obtain the reactions, shear forces, bending moments, slopes, and deflections of the beam. Construct the shear-force and bending-moment diagrams, labeling all critical ordinates.



Solution 10.3-2 Fixed-end beam (uniform load)

Select M_A as the redundant reaction.

REACTIONS (FROM SYMMETRY AND EQUILIBRIUM)

$$R_A = R_B = \frac{qL}{2} \qquad M_B = M_A$$

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A - \frac{qx^2}{2} = -M_A + \frac{q}{2}(Lx - x^2) \qquad (1)$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = -M_A + \frac{q}{2}(Lx - x^2)$$

$$EIv' = -M_A x + \frac{q}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_1 \qquad (2)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

$$EIv = -\frac{M_A x^2}{2} + \frac{q}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_2 \qquad (3)$$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $v(L) = 0 \quad \therefore M_A = \dfrac{qL^2}{12}$

REACTIONS

$$R_A = R_B = \frac{qL}{2} \quad M_A = M_B = \frac{qL^2}{12} \quad \longleftarrow$$

SHEAR FORCE (FROM EQUILIBRIUM)

$$V = R_A - qx = \frac{q}{2}(L - 2x) \quad \longleftarrow$$

BENDING MOMENT (FROM EQ. 1)

$$M = -\frac{q}{12}(L^2 - 6Lx + 6x^2) \quad \longleftarrow$$

SLOPE (FROM EQ. 2)

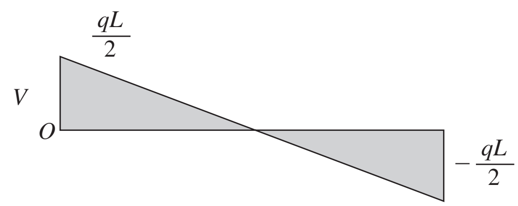
$$v' = -\frac{qx}{12EI}(L^2 - 3Lx + 2x^2) \quad \longleftarrow$$

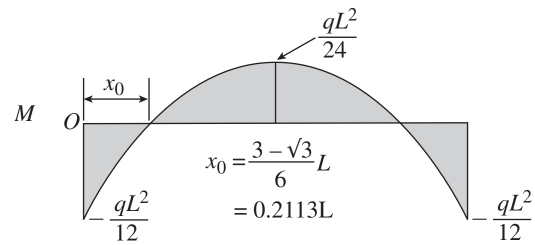
DEFLECTION (FROM EQ. 3)

$$v = -\frac{qx^2}{24EI}(L - x)^2 \quad \longleftarrow$$

$$\delta_{max} = -v\left(\frac{L}{2}\right) = \frac{qL^4}{384EI}$$

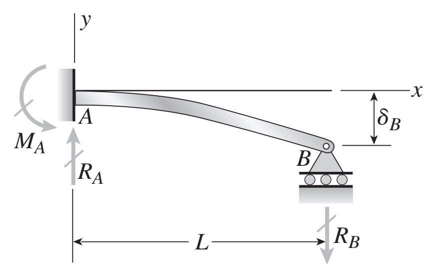
SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS





Problem 10.3-3 A cantilever beam AB of length L has a fixed support at A and a roller support at B (see figure). The support at B is moved downward through a distance δ_B.

Using the fourth-order differential equation of the deflection curve (the load equation), determine the reactions of the beam and the equation of the deflection curve. (*Note:* Express all results in terms of the imposed displacement δ_B.)



Solution 10.3-3 **Cantilever beam with imposed displacement δ_B**

REACTIONS (FROM EQUILIBRIUM)

$R_A = R_B$ (1) $\qquad M_A = R_B L$ (2)

DIFFERENTIAL EQUATIONS

$EIv'''' = -q = 0$ (3)

$EIv''' = V = C_1$ (4)

$EIv'' = M = C_1x + C_2$ (5)

$EIv' = C_1x^2/2 + C_2x + C_3$ (6)

$EIv = C_1x^3/6 + C_2x^2/2 + C_3x + C_4$ (7)

B.C. 1 $\quad v(0) = 0 \quad \therefore C_4 = 0$

B.C. 2 $\quad v'(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $\quad v''(L) = 0 \quad \therefore C_1L + C_2 = 0$ (8)

B.C. 4 $\quad v(L) = -\delta_B \quad \therefore C_1L + 3C_2 = -6EI\delta_B/L^2$ (9)

SOLVE EQUATIONS (8) AND (9):

$$C_1 = \frac{3EI\delta_B}{L^3} \qquad C_2 = -\frac{3EI\delta_B}{L^2}$$

SHEAR FORCE (EQ. 4)

$$V = \frac{3EI\delta_B}{L^3} \qquad R_A = V(0) = \frac{3EI\delta_B}{L^3}$$

REACTIONS (EQS. 1 AND 2)

$$R_A = R_B = \frac{3EI\delta_B}{L^3} \qquad M_A = R_BL = \frac{3EI\delta_B}{L^2} \quad \longleftarrow$$

DEFLECTION (FROM EQ. 7):

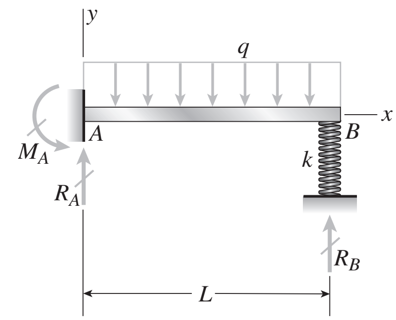
$$v = -\frac{\delta_B x^2}{2L^3}(3L - x) \quad \longleftarrow$$

SLOPE (FROM EQ. 6):

$$v' = -\frac{3\delta_B x}{2L^3}(2L - x)$$

Problem 10.3-4 A cantilever beam AB of length L has a fixed support at A and a spring support at B (see figure). The spring behaves in a linearly elastic manner with stiffness k.

If a uniform load of intensity q acts on the beam, what is the downward displacement δ_B of end B of the beam? (Use the second-order differential equation of the deflection curve, that is, the bending-moment equation.)



Solution 10.3-4 Beam with spring support

q = intensity of uniform load

EQUILIBRIUM $\quad R_A = qL - R_B \qquad (1)$

$$M_A = \frac{qL^2}{2} - R_B L \qquad (2)$$

SPRING $\quad R_B = k\delta_B \qquad (3)$

δ_B = downward displacement of point B.

BENDING MOMENT (FROM EQUILIBRIUM)

$$M = R_A x - M_A - \frac{qx^2}{2}$$

DIFFERENTIAL EQUATIONS

$$EIv'' = M = R_A x - M_A - \frac{qx^2}{2}$$

$$EIv' = R_A \frac{x^2}{2} - M_A x - \frac{qx^3}{6} + C_1$$

$$EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} - \frac{qx^4}{24} + C_1 x + C_2$$

B.C. 1 $\quad v'(0) = 0 \quad \therefore C_1 = 0$

B.C. 2 $\quad v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $\quad v(L) = -\delta_B$

$$\therefore -EI\delta_B = \frac{R_A L^3}{6} - \frac{M_A L^2}{2} - \frac{qL^4}{24}$$

Substitute R_A and M_A from Eqs. (1) and (2):

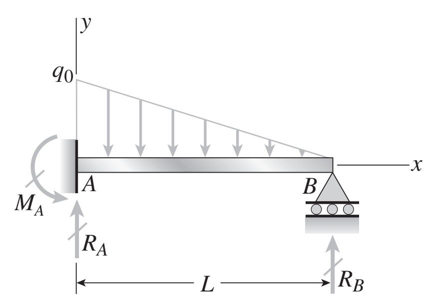
$$-EI\delta_B = \frac{R_B L^3}{3} - \frac{qL^4}{8}$$

Substitute for R_B from Eq. (3) and solve:

$$\delta_B = \frac{3qL^4}{24EI + 8kL^3} \quad \longleftarrow$$

Problem 10.3-5 A propped cantilever beam AB of length L supports a triangularly distributed load of maximum intensity q_0 (see figure).

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-5 Propped cantilever beam

Triangular load $\quad q = q_0(L - x)/L$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -\frac{q_0}{L}(L - x) \qquad (1)$$

$$EIv''' = V = -q_0 x + \frac{q_0 x^2}{2L} + C_1 \qquad (2)$$

$$EIv'' = M = -\frac{q_0 x^2}{2} + \frac{q_0 x^3}{6L} + C_1 x + C_2 \qquad (3)$$

$$EIv' = -\frac{q_0 x^3}{6} + \frac{q_0 x^4}{24L} + C_1 \frac{x^2}{2} + C_2 x + C_3 \qquad (4)$$

$$EIv = -\frac{q_0 x^4}{24} + \frac{q_0 x^5}{120L} + C_1 \frac{x^3}{6} + C_2 \frac{x^2}{2} + C_3 x + C_4 \qquad (5)$$

B.C. 1 $v''(L) = 0$ $\quad \therefore C_1L + C_2 = \dfrac{q_0L^2}{3}$ (6)

B.C. 2 $v'(0) = 0$ $\quad \therefore C_3 = 0$

B.C. 3 $v(0) = 0$ $\quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0$ $\quad \therefore C_1L + 3C_2 = \dfrac{q_0L^2}{5}$ (7)

Solve Eqs. (6) and (7):

$$C_1 = \frac{2q_0L}{5} \qquad C_2 = -\frac{q_0L^2}{15}$$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0}{10L}(4L^2 - 10Lx + 5x^2)$$

REACTIONS $R_A = V(0) = \dfrac{2q_0L}{5}$ ←

$R_B = -V(L) = \dfrac{q_0L}{10}$ ←

From equilibrium:

$$M_A = \frac{q_0L^2}{6} - R_BL = \frac{q_0L^2}{15} \leftarrow$$

DEFLECTION CURVE (FROM EQ. 5)

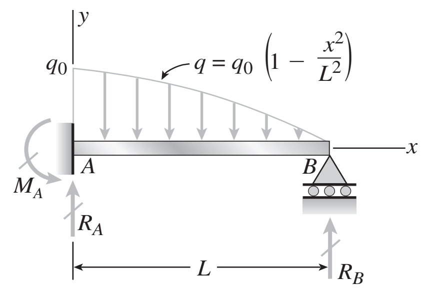
$$EIv = -\frac{q_0x^4}{24} + \frac{q_0x^5}{120L} + \frac{2q_0L}{5}\left(\frac{x^3}{6}\right) - \frac{q_0L^2}{15}\left(\frac{x^2}{2}\right)$$

or

$$v = -\frac{q_0x^2}{120LEI}(4L^3 - 8L^2x + 5Lx^2 - x^3) \leftarrow$$

Problem 10.3-6 The load on a propped cantilever beam AB of length L is parabolically distributed according to the equation $q = q_0(1 - x^2/L^2)$, as shown in the figure.

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-6 Propped cantilever beam

Parabolic load $q = q_0(1 - x^2/L^2)$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0(1 - x^2/L^2) \quad (1)$$

$$EIv''' = V = -q_0(x - x^3/3L^2) + C_1 \quad (2)$$

$$EIv'' = M = -q_0\left(\frac{x^2}{2} - \frac{x^4}{12L^2}\right) + C_1x + C_2 \quad (3)$$

$$EIv' = -q_0\left(\frac{x^3}{6} - \frac{x^5}{60L^2}\right) + C_1\frac{x^2}{2} + C_2x + C_3 \quad (4)$$

$$EIv = -q_0\left(\frac{x^4}{24} - \frac{x^6}{360L^2}\right) + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3x + C_4 \quad (5)$$

B.C. 1 $v''(L) = 0$ $\quad \therefore C_1L + C_2 = 5q_0L^2/12$ (6)

B.C. 2 $v'(0) = 0$ $\quad \therefore C_3 = 0$

B.C. 3 $v(0) = 0$ $\quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0$ $\quad \therefore C_1L + 3C_2 = 7q_0L^2/30$ (7)

Solve Eqs. (6) and (7):

$C_1 = 61q_0L/120 \qquad C_2 = -11q_0L^2/120$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0}{120L^2}(61L^3 - 120L^2x + 40x^3)$$

REACTIONS $R_A = V(0) = 61q_0L/120$ ←

$R_B = -V(L) = 19q_0L/120$ ←

From equilibrium:

$$M_A = \frac{2}{3}(q_0)(L)\left(\frac{3L}{8}\right) - R_BL = \frac{11q_0L^2}{120} \leftarrow$$

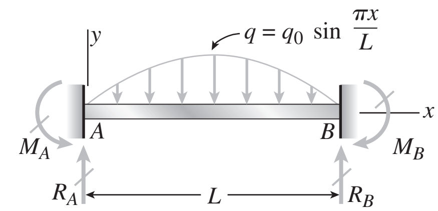
DEFLECTION CURVE (FROM EQ. 5)

$$v = -\frac{q_0x^2}{720L^2EI}(33L^4 - 61L^3x + 30L^2x^2 - 2x^4)$$

$$= -\frac{q_0x^2(L - x)}{720L^2EI}(33L^3 - 28L^2x + 2Lx^2 + 2x^3) \leftarrow$$

Problem 10.3-7 The load on a fixed-end beam *AB* of length *L* is distributed in the form of a sine curve (see figure). The intensity of the distributed load is given by the equation $q = q_0 \sin \pi x/L$.

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-7 Fixed-end beam (sine load)

$q = q_0 \sin \pi x/L$

FROM SYMMETRY: $R_A = R_B \qquad M_A = M_B$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0 \sin \pi x/L \qquad (1)$$

$$EIv''' = V = \frac{q_0 L}{\pi}\cos\frac{\pi x}{L} + C_1 \qquad (2)$$

$$EIv'' = M = \frac{q_0 L^2}{\pi^2}\sin\frac{\pi x}{L} + C_1 x + C_2 \qquad (3)$$

$$EIv' = -\frac{q_0 L^3}{\pi^3}\cos\frac{\pi x}{L} + C_1\frac{x^2}{2} + C_2 x + C_3 \qquad (4)$$

$$EIv = -\frac{q_0 L^4}{\pi^4}\sin\frac{\pi x}{L} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4 \qquad (5)$$

B.C. 1 From symmetry, $V\left(\frac{L}{2}\right) = 0 \quad \therefore C_1 = 0$

B.C. 2 $v'(0) = 0 \qquad \therefore C_3 = q_0 L^3/\pi^3$

B.C. 3 $v'(L) = 0 \qquad \therefore C_2 = -2q_0 L^2/\pi^3$

B.C. 4 $v(0) = 0 \qquad \therefore C_4 = 0$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0 L}{\pi}\cos\frac{\pi x}{L} \qquad R_A = V(0) = \frac{q_0 L}{\pi} \longleftarrow$$

$$R_B = R_A = \frac{q_0 L}{\pi} \longleftarrow$$

BENDING MOMENT (EQ. 3)

$$M = \frac{q_0 L^2}{\pi^3}\left(\pi \sin\frac{\pi x}{L} - 2\right)$$

$$M_A = -M(0) = \frac{2q_0 L^2}{\pi^3} \qquad M_B = M_A = \frac{2q_0 L^2}{\pi^3} \longleftarrow$$

DEFLECTION CURVE (FROM EQ. 5)

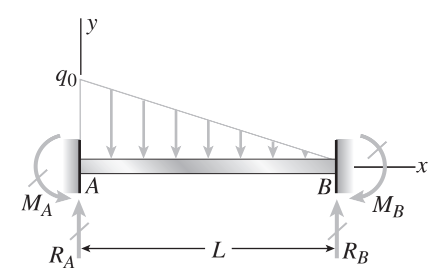
$$EIv = -\frac{q_0 L^4}{\pi^4}\sin\frac{\pi x}{L} - \frac{q_0 L^2 x^2}{\pi^3} + \frac{q_0 L^3 x}{\pi^3}$$

or

$$v = -\frac{q_0 L^2}{\pi^4 EI}\left(L^2 \sin\frac{\pi x}{L} + \pi x^2 - \pi L x\right) \longleftarrow$$

Problem 10.3-8 A fixed-end beam *AB* of length *L* supports a triangularly distributed load of maximum intensity q_0 (see figure).

Beginning with the fourth-order differential equation of the deflection curve (the load equation), obtain the reactions of the beam and the equation of the deflection curve.



Solution 10.3-8 Fixed-end beam (triangular load)

$q = q_0(1 - x/L)$

DIFFERENTIAL EQUATIONS

$$EIv'''' = -q = -q_0\left(1 - \frac{x}{L}\right) \qquad (1)$$

$$EIv''' = V = -q_0\left(x - \frac{x^2}{2L}\right) + C_1 \qquad (2)$$

$$EIv'' = M = -q_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + C_1 x + C_2 \qquad (3)$$

$$EIv' = -q_0\left(\frac{x^3}{6} - \frac{x^4}{24L}\right) + C_1\frac{x^2}{2} + C_2 x + C_3 \qquad (4)$$

$$EIv = -q_0\left(\frac{x^4}{24} - \frac{x^5}{120L}\right) + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4 \qquad (5)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_3 = 0$

B.C. 2 $v'(L) = 0 \quad \therefore C_1 L + 2C_2 = \frac{q_0 L^2}{4}$ (6)

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_1 L + 3C_2 = \frac{q_0 L^2}{5}$ (7)

Solve eqs. (6) and (7):

$$C_1 = \frac{7q_0 L}{20} \qquad C_2 = -\frac{q_0 L^2}{20}$$

SHEAR FORCE (EQ. 2)

$$V = \frac{q_0}{20L}(7L^2 - 20Lx + 10x^2)$$

REACTIONS $R_A = V(0) = \frac{7q_0 L}{20}$ ←

$R_B = -V(L) = \frac{3q_0 L}{20}$ ←

BENDING MOMENT (EQ. 3)

$$M = -\frac{q_0}{60L}(3L^3 - 21L^2x + 30Lx^2 - 10x^3)$$

REACTIONS $M_A = -M(0) = \frac{q_0 L^2}{20}$ ←

$M_B = -M(L) = \frac{q_0 L^2}{30}$ ←

DEFLECTION CURVE (EQ. 5)

$$v = -\frac{q_0 x^2}{120LEI}(3L^3 - 7L^2x + 5Lx^2 - x^3)$$

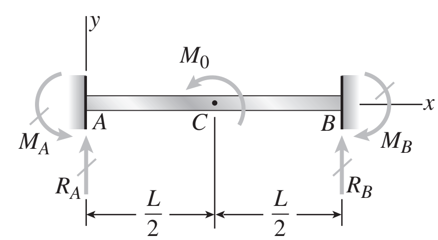
or

$$v = -\frac{q_0 x^2}{120LEI}(L - x)^2(3L - x)$$ ←

Problem 10.3-9 A counterclockwise moment M_0 acts at the midpoint of a fixed-end beam ACB of length L (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), determine all reactions of the beam and obtain the equation of the deflection curve for the left-hand half of the beam.

Then construct the shear-force and bending-moment diagrams for the entire beam, labeling all critical ordinates. Also, draw the deflection curve for the entire beam.



Solution 10.3-9 Fixed-end beam (M_0 = applied load)

Beam is symmetric; load is antisymmetric.

Therefore, $R_A = -R_B \quad M_A = -M_B \quad \delta_C = 0$

DIFFERENTIAL EQUATIONS ($0 \le x \le L/2$)

$$EIv'' = M = R_A x - M_A \qquad (1)$$

$$EIv' = R_A \frac{x^2}{2} - M_A x + C_1 \qquad (2)$$

$$EIv = R_A \frac{x^3}{6} - M_A \frac{x^2}{2} + C_1 x + C_2 \qquad (3)$$

B.C. 1 $v'(0) = 0 \quad \therefore C_1 = 0$

B.C. 2 $v(0) = 0 \quad \therefore C_2 = 0$

B.C. 3 $v\left(\frac{L}{2}\right) = 0 \quad \therefore M_A = \frac{R_A L}{6}$ Also, $M_B = \frac{-R_A L}{6}$

EQUILIBRIUM (OF ENTIRE BEAM)

$\sum M_B = 0 \qquad M_A + M_0 - M_B - R_A L = 0$

or, $\frac{R_A L}{6} + M_0 + \frac{R_A L}{6} - R_A L = 0$

$\therefore R_A = -R_B = \frac{3M_0}{2L}$ ←

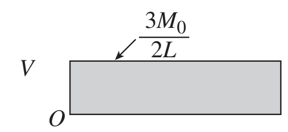
$M_A = \frac{R_A L}{6} \qquad \therefore M_A = -M_B = \frac{M_0}{4}$ ←

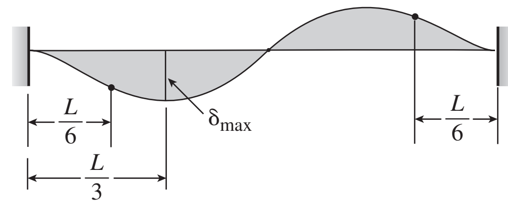
DEFLECTION CURVE (EQ. 3)

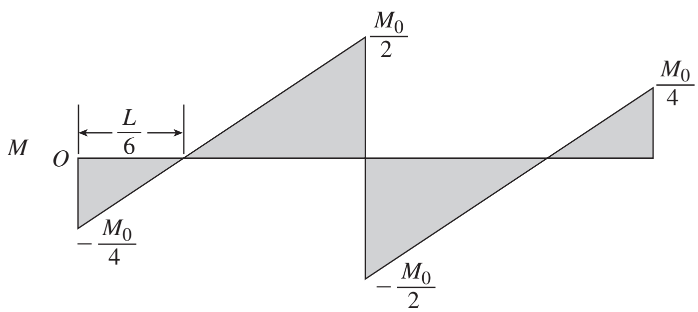
$$v = -\frac{M_0 x^2}{8LEI}(L - 2x) \quad \left(0 \le x \le \frac{L}{2}\right)$$ ←

(Continued)

DIAGRAMS







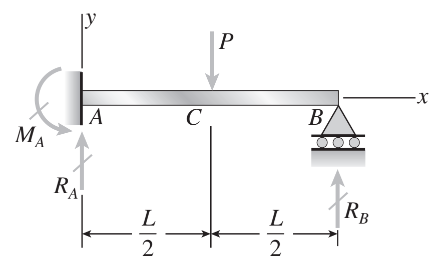
$$\delta_{max} = \frac{M_0 L^2}{216\,EI}$$

At point of inflection: $\delta = \delta_{max}/2$

Problem 10.3-10 A propped cantilever beam AB supports a concentrated load P acting at the midpoint C (see figure).

Beginning with the second-order differential equation of the deflection curve (the bending-moment equation), determine all reactions of the beam and draw the shear-force and bending-moment diagrams for the entire beam.

Also, obtain the equations of the deflection curves for both halves of the beam, and draw the deflection curve for the entire beam.



Solution 10.3-10 Propped cantilever beam

P = applied load at $x = L/2$

Select R_B as redundant reaction.

REACTIONS (FROM EQUILIBRIUM)

$$R_A = P - R_B \qquad (1) \qquad\qquad M_A = \frac{PL}{2} - R_B L \qquad (2)$$

BENDING MOMENTS (FROM EQUILIBRIUM)

$$M = R_A x - M_A = (P - R_B)x - \left(\frac{PL}{2} - R_B L\right) \qquad \left(0 \le x \le \frac{L}{2}\right)$$

$$M = R_B(L - x) \qquad \left(\frac{L}{2} \le x \le L\right)$$

DIFFERENTIAL EQUATIONS $(0 \le x \le L/2)$

$$EIv'' = M = (P - R_B)\,x - \left(\frac{PL}{2} - R_B L\right) \qquad (3)$$

$$EIv' = (P - R_B)\frac{x^2}{2} - \left(\frac{PL}{2} - R_B L\right)x + C_1 \qquad (4)$$

$$EIv = (P - R_B)\frac{x^3}{6} - \left(\frac{PL}{2} - R_B L\right)\frac{x^2}{2} + C_1 x + C_2 \qquad (5)$$

B.C. 1 $\quad v'(0) = 0 \qquad \therefore C_1 = 0$

B.C. 2 $\quad v(0) = 0 \qquad \therefore C_2 = 0$

DIFFERENTIAL EQUATIONS $(L/2 \le x \le L)$

$$EIv'' = M = R_B(L - x) \qquad (6)$$

$$EIv' = R_B L x - R_B\frac{x^2}{2} + C_3 \qquad (7)$$

$$EIv = R_B L\frac{x^2}{2} - R_B\frac{x^3}{6} + C_3 x + C_4 \qquad (8)$$

B.C. 3 $\quad v(L) = 0 \qquad \therefore C_3 L + C_4 = -\dfrac{R_B L^3}{3} \qquad (9)$

B.C. 4 Continuity condition at point C

At $x = \dfrac{L}{2}$: $\quad (v')_{\text{Left}} = (v')_{\text{Right}}$

$$(P - R_B)\left(\frac{L^2}{8}\right) - \left(\frac{PL}{2} - R_B L\right)\left(\frac{L}{2}\right)$$

$$= R_B L\left(\frac{L}{2}\right) - R_B\left(\frac{L^2}{8}\right) + C_3$$

$$\text{or } C_3 = -\frac{PL^2}{8} \qquad (10)$$

$$\text{From Eq. (9): } C_4 = -\frac{R_B L^3}{3} + \frac{PL^3}{8} \qquad (11)$$

B.C. 5 Continuity condition at point C.

At $x = \dfrac{L}{2}$: $\quad (v)_{\text{Left}} = (v)_{\text{Right}}$

$$(P - R_B)\frac{L^3}{48} - \left(\frac{PL}{2} - R_B L\right)\frac{L^2}{8}$$

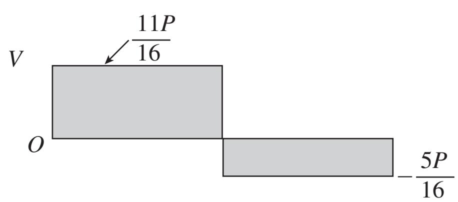
$$= R_B L\left(\frac{L^2}{8}\right) - R_B\left(\frac{L^3}{48}\right) - \frac{PL^2}{8}\left(\frac{L}{2}\right) - \frac{R_B L^3}{3} + \frac{PL^3}{8}$$

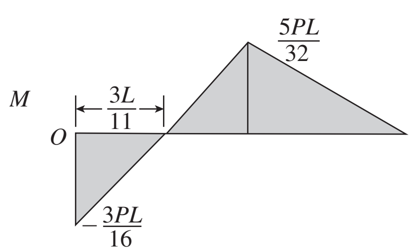
$$\text{or } R_B = \frac{5P}{16} \quad \longleftarrow$$

$$\text{From eq. (1): } \quad R_A = P - R_B = \frac{11P}{16} \quad \longleftarrow$$

$$\text{From eq. (2): } \quad M_A = \frac{PL}{2} - R_B L = \frac{3PL}{16} \quad \longleftarrow$$

SHEAR-FORCE AND BENDING MOMENT DIAGRAMS





DEFLECTION CURVE FOR $0 \le x \le L/2$ (FROM EQ. 5)

$$v = -\frac{Px^2}{96EI}(9L - 11x) \quad (0 \le x \le L/2) \quad \longleftarrow$$

DEFLECTION CURVE FOR $L/2 \le x \le L$ (FROM EQ. 8)

$$v = -\frac{P}{96EI}(-2L^3 + 12L^2x - 15Lx^2 + 5x^3)$$

$$= -\frac{P}{96EI}(L - x)(-2L^2 + 10Lx - 5x^2)$$

$$(L/2 \le x \le L) \quad \longleftarrow$$

SLOPE IN RIGHT-HAND PART OF THE BEAM

$$\text{From eq. (7): } \quad v' = -\frac{P}{32EI}(4L^2 - 10Lx + 5x^2)$$

Point of zero slope:

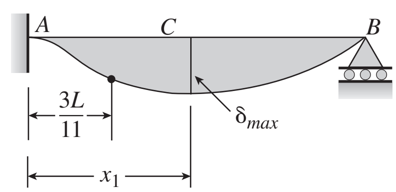
$$5x_1^2 - 10Lx_1 + 4L^2 = 0 \quad x_1 = \frac{L}{5}\left(5 - \sqrt{5}\right)$$

$$= 0.5528L$$

MAXIMUM DEFLECTION

$$\delta_{\max} = -(v)_{x=x_1} = 0.009317\frac{PL^3}{EI}$$

DEFLECTION CURVE

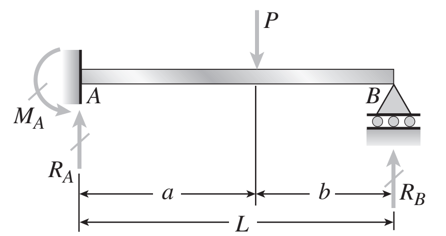


Method of Superposition

The problems for Section 10.4 are to be solved by the method of superposition. All beams have constant flexural rigidity EI unless otherwise stated. When drawing shear-force and bending-moment diagrams, be sure to label all critical ordinates, including maximum and minimum values.

Problem 10.4-1 A propped cantilever beam AB of length L carries a concentrated load P acting at the position shown in the figure.

Determine the reactions R_A, R_B, and M_A for this beam. Also, draw the shear-force and bending-moment diagrams, labeling all critical ordinates.



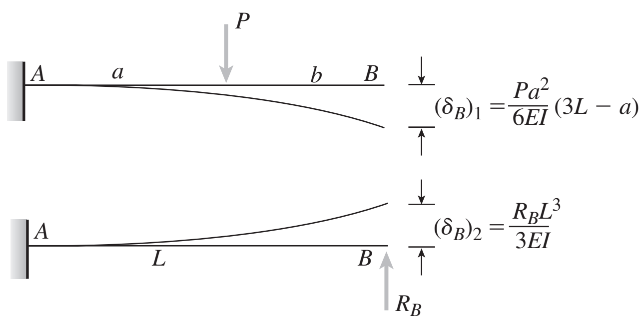
Solution 10.4-1 Propped cantilever beam

Select R_B as redundant.

EQUILIBRIUM

$$R_A = P - R_B \qquad M_A = Pa - R_B L$$

RELEASED STRUCTURE AND FORCE-DISPLACEMENT RELATIONS



COMPATIBILITY

$$\delta_B = (\delta_B)_1 - (\delta_B)_2 = 0$$

$$\delta_B = \frac{Pa^2}{6EI}(3L - a) - \frac{R_B L^3}{3EI} = 0$$

$$R_B = \frac{Pa^2}{2L^3}(3L - a) \quad \longleftarrow$$

OTHER REACTIONS (FROM EQUILIBRIUM)

$$R_A = \frac{Pb}{2L^3}(3L^2 - b^2) \quad M_A = \frac{Pab}{2L^2}(L + b) \quad \longleftarrow$$

SHEAR-FORCE AND BENDING-MOMENT DIAGRAMS